

Spitzer identity, Wiener-Hopf factorization and pricing of discretely monitored exotic options

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Abstract

The Wiener-Hopf factorization of a complex function arises in a variety of fields in applied mathematics such as probability, finance, insurance, queuing theory, radio engineering and fluid mechanics. Its importance is that it fully characterizes the distribution of functionals of a random walk or a Lévy process, such as the maximum, the minimum and hitting times. Here we propose a constructive procedure, valid for both single and double barriers, based on the combined use of the Hilbert and the z -transform. The numerical implementation can be simply performed via the fast Fourier transform and the Euler summation. Given that the information in the Wiener-Hopf factors is strictly related to the distributions of the first passage times, as a concrete application in mathematical finance we consider the pricing of discretely monitored exotic options, such as lookback and barrier options, when the underlying price evolves according to an exponential Lévy process. We show that the computational cost of our procedure is independent of the number of monitoring dates and the error decays exponentially with the number of

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grid points.

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1. Introduction

This paper provides a new procedure to determine the finite-time distribution of the discrete extrema and of the hitting times of one or two barriers for a process with independent and identically distributed increments, such as a Lévy process. Spitzer [48] provided a closed formula for the z -transform (or moment generating function or discrete Laplace transform) of the characteristic function of the extrema of a random walk observed on a set of discrete dates. Up to now the concrete application of the Spitzer identity has been difficult because it requires the Wiener-Hopf (WH) factorization of a function defined in the complex plane, a mathematical problem that concerns a variety of fields in applied mathematics. Indeed, this factorization cannot be achieved analytically except in few cases, or its computation turns out to be very demanding requiring the numerical evaluation of a multidimensional integral in the complex plane. In addition, with regard to a general Lévy process, little is known for the two-barriers case.

The key contributions of our paper are the following. First of all, we provide a constructive procedure for performing the Wiener-Hopf factorization. More precisely, we express the Wiener-Hopf factors arising in the Spitzer identity in terms of the Plemelj-Sokhotsky relations, which allow to compute the Wiener-Hopf factors through the Hilbert transform. The latter is then approximated via a sinc function expansion [49], which guarantees an exponential decay of the approximation error on the number of grid points.

Moreover, our methodology can deal with both a single and a double barrier. The solution in the second case is of interest in itself because it solves a long-standing problem related to an efficient computation of the Wiener-Hopf factors in the presence of two barriers. The double-barrier case did not admit a simple feasible solution up to now, except under few special assumptions on the structure of the Lévy process. One has to solve two coupled integral equations, which can be achieved factorizing a 2×2 matrix of functions, but a general analytical method for this more difficult problem has not been found yet [29]. Here, as the second main contribution of the paper, we constructively propose a fixed-point algorithm based on an extension of the single-barrier case that achieves a fast convergence.

As a concrete application, we contribute to the mathematical finance literature related to the pricing of exotic options, such as barrier and lookback. The application of transform techniques in mathematical finance is rather recent. The first and most important contributions are probably the articles by Heston [26] and Carr and Madan [12], where the authors show how to price European options with non-Gaussian models exploiting the Fourier transform. Similar techniques were developed later for path-dependent derivatives [e.g. 10, 16, 23]. Our paper provides a unified framework and a fast operational method for pricing barrier and lookback (or hindsight) options when the underlying asset evolves as an exponential Lévy process. In addition, the monitoring condition, e.g., the event that the underlying asset value falls below a given barrier for a down-and-out barrier option, is assumed to be controlled at discrete time intervals. Our procedure, based on the new WH factorization method, has a computational cost independent of the number of monitoring dates. This is possible because the inversion of the discrete Laplace transform is performed via the Euler acceleration, which bounds from above the number of Wiener-Hopf factorizations to be computed. Moreover, at least with regard to single-barrier and lookback options, the method provides exponential order of convergence due to the fact that the factorization is performed remaining in the complex plane. The existing pricing methods are based on the backward recursive formula [e.g. 19, 20, 21, 27, 42], and on exploiting the convolution structure of the transition density of the Lévy process by performing the computations efficiently and fast using the FFT, which leads to a CPU time that grows as $\mathcal{O}(M \log M)$, where M is the number of discretization points. However, all the above cited methods are characterized by a polynomial decay of the error with M . This order of accuracy is related to the fact that the backward procedure for barrier options involves a convolution, that can be computed in the complex plane, and a projection, which is applied in the real plane, to take into account the presence of the barrier. A noticeable exception was presented by Feng and Linetsky [16, 17], who reformulated the backward procedure for barrier and lookback options in terms of the Hilbert transform, so that all steps are performed in the complex plane. Computing the Hilbert transform with a sinc function expansion, they achieved an exponential decay of the error. However, the computational cost of all these methods, including the one by Feng and Linetsky, increases linearly with the number of monitoring dates.

Finally, the factorization procedure introduced here is quite general and can also be applied, without any additional complication, to continuously-monitored contracts. Even the best available method listed above, i.e., that by Feng and Linetsky, does not have this feature.

Even if the Spitzer identity has already been used in option pricing [e.g.

6, 7, 23, 40] and the present paper is mainly focused on this kind of applications, our method goes well beyond option pricing and opens up the way to a more extensive use of the Spitzer identity and the Wiener-Hopf factorization in a variety of non-financial applications. In this regard we would like to mention the applicability to queuing theory due to the strict connection between random walks and queues, see Lindley [41] for pioneering contributions as well as Cohen [13], Prabhu [43] and Asmussen [2, 3]. Further applications include insurance [22] and sequential testing [47]. Finally, the Wiener-Hopf factorization arises in many branches of engineering, mathematical physics and applied mathematics. This is testified by the thousands of papers published on the subject since its conception. A review of the different applications is given by Lawrie and Abrahams [37].

The structure of the paper is the following. Section 2 introduces the Spitzer identity and its relationship with the Wiener-Hopf factorization, proposing, via the interpretation of the Plemelj-Sokhotsky relations as Hilbert transforms, a new operational method to perform the factorization and therefore to compute the distributions of the minimum and the maximum of a Lévy process, as well as the joint distributions of the process at maturity and of its minimum or maximum over the whole time interval. Section 3 shows how the proposed general methodology can be implemented efficiently and accurately computing the Hilbert transform via a sinc expansion; we also discuss the inversion of the z -transform and its acceleration through the Euler summation rule to make the computational cost independent of the number of monitoring dates. Section 4 deals with the pricing problem for lookback and barrier options, describing how our procedure is fast as well as accurate. This is validated numerically in Section 5 with a variety of numerical experiments.

2. Spitzer identity and Wiener-Hopf factorization

We consider a Lévy process $X(t)$, i.e., a stochastic process with $X(0) = 0$ and independent and identically distributed increments. The Lévy-Khincine formula states that the characteristic function of the process is given by $\Psi(\xi, t) = \mathbb{E}[e^{i\xi X(t)}] = e^{\psi(\xi)t}$, where ψ is the characteristic exponent of the process,

$$\psi(\xi) = ia\xi - \frac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}} (e^{i\xi\eta} - 1 - i\xi\eta\mathbf{1}_{|\eta|<1}) \nu(d\eta); \quad (1)$$

the parameters (a, σ, ν) are the Lévy-Khincine triplet which fully defines the Lévy process $X(t)$.

In several applications in queueing theory, insurance and financial mathematics, the key point is the determination of the law of the extrema of the

Lévy process observed on an equally-spaced grid $X_n = X(n\Delta)$, $n = 0, \dots, N$, where $\Delta > 0$ is the time step, i.e., the distance between two consecutive monitoring dates, which is assumed constant. We define the processes of the maximum M_N and of the minimum m_N up to the N th monitoring date as

$$M_N = \max_{n=0, \dots, N} X_n \quad \text{and} \quad m_N = \min_{n=0, \dots, N} X_n. \quad (2)$$

To distinguish the present case, where the above processes, albeit evolving in continuous time, are recorded only at discrete times, the terminology discrete versus continuous monitoring is used.

In particular, besides the distribution $P_X(x, N)$ of the Lévy process at maturity $T = N\Delta$, we will need the distributions $P_m(x, N)$ of the minimum and $P_M(x, N)$ of the maximum over the whole set $\{n = 0, \dots, N\}$, as well as the joint distributions $P_{X,m}(x, N)$ or $P_{X,M}(x, N)$ of the process at maturity and of its minimum or maximum over the interval with respect to a lower (upper) barrier l (u), and the joint distribution of the triplet (X_N, m_N, M_N) , $P_{X,m,M}(x, N)$. These distributions are defined as

$$dP_X(x, N) = p_X(x, N)dx = \mathbb{P}[X_N \in [x, x + dx)] \quad (3)$$

$$dP_m(x, N) = p_m(x, N)dx = \mathbb{P}[m_N \in [x, x + dx)] \quad (4)$$

$$dP_M(x, N) = p_M(x, N)dx = \mathbb{P}[M_N \in [x, x + dx)] \quad (5)$$

$$dP_{X,m}(x, N) = p_{X,m}(x, N)dx = \mathbb{P}[X_N \in [x, x + dx) \cap m_N > l] \quad (6)$$

$$dP_{X,M}(x, N) = p_{X,M}(x, N)dx = \mathbb{P}[X_N \in [x, x + dx) \cap M_N < u] \quad (7)$$

$$dP_{X,m,M}(x, N) = p_{X,m,M}(x, N)dx = \mathbb{P}[X_N \in [x, x + dx) \cap m_N > l \cap M_N < u]. \quad (8)$$

We define the Fourier transform of a function $g(x)$ as

$$\widehat{g}(\xi) = \mathcal{F}_{x \rightarrow \xi}[g(x)](\xi) := \int_{-\infty}^{+\infty} g(x) e^{i\xi x} dx$$

and its inverse with

$$g(x) = \mathcal{F}_{\xi \rightarrow x}^{-1}[\widehat{g}(\xi)](x) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{g}(\xi) e^{-ix\xi} d\xi.$$

When no misunderstanding is possible about which variable is being Fourier-transformed, notably when the argument function depends on a single variable, we will drop the subscript to the operators \mathcal{F} and \mathcal{F}^{-1} . In some cases, for compatibility with previous literature we use an upper-case letter instead of a lower-case letter with a hat, i.e., $G(\xi)$ instead of $\widehat{g}(\xi)$. As an exception

to these notations, the above defined characteristic function Ψ of the Lévy process is the Fourier transform of the probability density function f of the Lévy process,

$$\Psi(\xi, \Delta) = \mathcal{F}_{x \rightarrow \xi}[f(x, \Delta)](\xi, \Delta),$$

where the transition probability that $X(t + \Delta) = x$ when $X(t) = x'$ has density $f(x - x', \Delta)$ for any $t > 0$. The convolution form of the density function is due to the assumption of independent increments.

Next, we define the z -transform (or generating function) of a discrete set of functions $v(x, n)$, $n \in \mathbb{N}_0$, as

$$\tilde{v}(x, q) = \mathcal{Z}_{n \rightarrow q}[v(x, n)](x, q) := \sum_{n=0}^{\infty} q^n v(x, n),$$

with $q \in \mathbb{C}$ (in the more common definition, z^{-1} is used in place of q). It is a discrete version of the Laplace transform. The original function $v(x, n)$ can be recovered through the complex integral

$$v(x, n) = \mathcal{Z}_{q \rightarrow n}^{-1}[\tilde{v}(x, q)](x, n) = \frac{1}{2\pi\rho^n} \int_0^{2\pi} \tilde{v}(x, \rho e^{iu}) e^{-inu} du, \quad (9)$$

where ρ must be within the radius of convergence [1].

Using combinatorial arguments, Spitzer [48] derived formulas for the z -transforms of the characteristic functions of the distributions defined in Equations (3)–(8), the celebrated Spitzer identities. We recall them here. Let Φ_{\pm} be two functions which are analytic in the overlap of two half planes including the real line such that

$$\Phi(\xi, q) := 1 - q\mathbb{E}[e^{i\xi X(\Delta)}] = 1 - q\Psi(\xi, \Delta) = \Phi_+(\xi, q)\Phi_-(\xi, q). \quad (10)$$

$\Phi_{\pm}(\xi, q)$ are the positive and negative Wiener-Hopf factors of $1 - q\Psi(\xi, \Delta)$.¹ The Spitzer identities express the desired characteristic functions through

¹The Wiener-Hopf factors are not uniquely defined: given a factorization $\Phi(\xi, q) = \Phi_+(\xi, q)\Phi_-(\xi, q)$, also $a\Phi_+(\xi, q)$ and $\frac{1}{a}\Phi_-(\xi, q)$ are Wiener-Hopf factors for any constant $a \neq 0$.

the inversion of a moment-generating function involving Φ , Φ_+ and Φ_- :

$$\tilde{p}_X(\xi, q) = \sum_{n=0}^{\infty} q^n \hat{p}_X(\xi, n) = \sum_{n=0}^{\infty} q^n \mathbb{E}(e^{i\xi X_n}) = \frac{1}{\Phi(\xi, q)} \quad (11)$$

$$\tilde{p}_m(\xi, q) = \sum_{n=0}^{\infty} q^n \hat{p}_m(\xi, n) = \sum_{n=0}^{\infty} q^n \mathbb{E}(e^{i\xi m_n}) = \frac{1}{\Phi_+(0, q)\Phi_-(\xi, q)} \quad (12)$$

$$\tilde{p}_M(\xi, q) = \sum_{n=0}^{\infty} q^n \hat{p}_M(\xi, n) = \sum_{n=0}^{\infty} q^n \mathbb{E}(e^{i\xi M_n}) = \frac{1}{\Phi_+(\xi, q)\Phi_-(0, q)} \quad (13)$$

$$\tilde{p}_{X,m}(\xi, q) = \sum_{n=0}^{\infty} q^n \hat{p}_{X,m}(\xi, n) = \frac{1}{\Phi(\xi, q)} - e^{il\xi} \frac{P_-(\xi, q)}{\Phi_+(\xi, q)} = e^{il\xi} \frac{P_+(\xi, q)}{\Phi_+(\xi, q)} \quad (14)$$

$$\tilde{p}_{X,M}(\xi, q) = \sum_{n=0}^{\infty} q^n \hat{p}_{X,M}(\xi, n) = \frac{1}{\Phi(\xi, q)} - e^{iu\xi} \frac{Q_+(\xi, q)}{\Phi_-(\xi, q)} = e^{iu\xi} \frac{Q_-(\xi, q)}{\Phi_-(\xi, q)} \quad (15)$$

$$\tilde{p}_{X,m,M}(\xi, q) = \sum_{n=0}^{\infty} q^n \hat{p}_{X,m,M}(\xi, n) = \frac{1}{\Phi(\xi, q)} - e^{il\xi} \frac{J_-(\xi, q)}{\Phi(\xi, q)} - e^{iu\xi} \frac{J_+(\xi, q)}{\Phi(\xi, q)}, \quad (16)$$

where

$$P(\xi, q) := \frac{e^{-il\xi}}{\Phi_-(\xi, q)} = P_+(\xi, q) + P_-(\xi, q)$$

and

$$Q(\xi, q) := \frac{e^{-iu\xi}}{\Phi_+(\xi, q)} = Q_+(\xi, q) + Q_-(\xi, q).$$

Notice that the joint probabilities in Equations (14)–(16) are given by the probability of the process at maturity, Equation (11), minus the probability to hit a barrier; the latter vanishes if the barrier moves to $\pm\infty$. Similar identities exist for the continuous-monitoring case too, where the quantity to be factorized is simply $\phi(\xi) = 1 - q\psi(\xi)$. Full technical details are given in Refs. [24, 36, 46].

The double-barrier problem, which is more difficult than the others, was not examined by Spitzer himself, but by Kemperman [30]. Unfortunately he did not present a constructive procedure for the determination of the quantities $J_+(\xi, q)$ and $J_-(\xi, q)$ in Equation (16). The problem was later solved in the Gaussian case by Green et al. [23, Section 2.4]. Here we generalize the latter construction to Lévy processes. In particular, Green et al. [23] proved

that $J_+(\xi, q)$ and $J_-(\xi, q)$ are the solution of the coupled integral equations

$$\frac{J_-(\xi, q)}{\Phi_-(\xi, q)} + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{i(u-l)\xi'} J_+(\xi', q)}{(\xi - \xi') \Phi_-(\xi', q)} d\xi' = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{-il\xi'}}{(\xi - \xi') \Phi_-(\xi', q)} d\xi', \quad (17)$$

$$\frac{J_+(\xi, q)}{\Phi_+(\xi, q)} + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{i(l-u)\xi'} J_-(\xi', q)}{(\xi - \xi') \Phi_+(\xi', q)} d\xi' = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{-iu\xi'}}{(\xi - \xi') \Phi_+(\xi', q)} d\xi', \quad (18)$$

where $\text{Im } \xi' > \text{Im } \xi$ in the first equation and $\text{Im } \xi' < \text{Im } \xi$ in the second (Im is the imaginary part).

As proved by Krein [33], the decomposition of a complex function $\widehat{f}(\xi) = \widehat{f}_+(\xi) + \widehat{f}_-(\xi)$ can be computed through the Cauchy-type integrals

$$\begin{aligned} \widehat{f}_+(\xi) &= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\widehat{f}(\xi')}{\xi - \xi'} d\xi', \quad \text{Im } \xi' < \text{Im } \xi, \\ \widehat{f}_-(\xi) &= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\widehat{f}(\xi')}{\xi - \xi'} d\xi', \quad \text{Im } \xi' > \text{Im } \xi. \end{aligned}$$

Therefore Equations (17)–(18) can be rewritten as

$$\begin{aligned} \frac{J_-(\xi, q)}{\Phi_-(\xi, q)} + \left[\frac{e^{i(u-l)\xi} J_+(\xi, q)}{\Phi_-(\xi, q)} \right]_- &= \left[\frac{e^{-il\xi}}{\Phi_-(\xi, q)} \right]_-, \\ \frac{J_+(\xi, q)}{\Phi_+(\xi, q)} + \left[\frac{e^{i(l-u)\xi} J_-(\xi, q)}{\Phi_+(\xi, q)} \right]_+ &= \left[\frac{e^{-iu\xi}}{\Phi_+(\xi, q)} \right]_+, \end{aligned}$$

or

$$\frac{J_-(\xi, q)}{\Phi_-(\xi, q)} = \left[\frac{e^{-il\xi} - e^{i(u-l)\xi} J_+(\xi, q)}{\Phi_-(\xi, q)} \right]_-, \quad (19)$$

$$\frac{J_+(\xi, q)}{\Phi_+(\xi, q)} = \left[\frac{e^{-iu\xi} - e^{i(l-u)\xi} J_-(\xi, q)}{\Phi_+(\xi, q)} \right]_+. \quad (20)$$

The solution of Equations (19)–(20) is related to the difficult problem of a matrix Wiener-Hopf factorization; a solution for the kind of matrix arising in this case has not been found yet. Therefore, we propose the following fixed-point algorithm that achieves a fast convergence: starting from a guess function $J_+^{(0)}(\xi, q) = 0$, for $j = 1, \dots$

(a) Decompose

$$\overline{P}^{(j)}(\xi, q) := \frac{e^{-il\xi}}{\Phi_-(\xi, q)} - \frac{e^{i(u-l)\xi} J_+^{(j-1)}(\xi, q)}{\Phi_-(\xi, q)} = \overline{P}_+^{(j)}(\xi, q) + \overline{P}_-^{(j)}(\xi, q) \quad (21)$$

and compute $J_-^{(j)}(\xi, q) = \overline{P}_-^{(j)}(\xi, q) \Phi_-(\xi, q)$.

(b) Decompose

$$\overline{Q}^{(j)}(\xi, q) := \frac{e^{-iu\xi}}{\Phi_+(\xi, q)} - \frac{e^{i(l-u)\xi} J_-^{(j)}(\xi, q)}{\Phi_+(\xi, q)} = \overline{Q}_+^{(j)}(\xi, q) + \overline{Q}_-^{(j)}(\xi, q) \quad (22)$$

and compute $J_+^{(j)}(\xi, q) = \overline{Q}_+^{(j)}(\xi, q)\Phi_+(\xi, q)$.

(c) If the distance between the new and old functions $J_\pm^{(j)}$ and $J_\pm^{(j-1)}$ is greater than a given tolerance, increase j and return to Step (a), otherwise stop and set $J_- = J_-^{(j)}$, $J_+ = J_+^{(j)}$.

The above fixed-point algorithm will be used to price double-barrier derivatives in Section 4.3.

To make all the above expressions usable, we need to factorize (or decompose) a complex function, defined in a strip containing the real axis, into a product (or sum) of two functions which are analytic in the overlap of two half planes, including the real line, where they are defined. Once this has been done and the relevant quantities in Equations (11)–(16) have been obtained, we must compute numerically an inverse z -transform, followed by an inverse Fourier transform. The latter is done in a standard way using the FFT. The inversion of the z -transform is rather easy too. It has been discussed in detail by Abate and Whitt [1], who showed that it can be well approximated by

$$v(x, n) = \mathcal{Z}_{q \rightarrow n}^{-1} \tilde{v}(x, q) \approx \frac{\tilde{v}(x, \rho) + 2 \sum_{j=1}^{n-1} (-1)^j \tilde{v}(x, \rho e^{ij\pi/n}) + (-1)^n \tilde{v}(x, -\rho)}{2n\rho^n}. \quad (23)$$

The more challenging part is the factorization of Φ in Equation (10), as well as the decomposition of P and Q ($\overline{P}^{(i)}$ and $\overline{Q}^{(i)}$ in the double-barrier case). In general, this problem can be described as follows. Given a smooth enough function $\widehat{f}(\xi)$, analytic in a strip around the real axis, we need to compute $\widehat{f}_\pm(\xi)$ such that

$$\widehat{f}(\xi) = \widehat{f}_+(\xi) \widehat{f}_-(\xi); \quad (24)$$

$\widehat{f}_+(\xi)$ is such that its inverse Fourier transform $f_+(x) = 0$ for $x < 0$, while $\widehat{f}_-(\xi)$ is such that $f_-(x) = 0$ for $x > 0$. Taking logarithms, this can be accomplished by the decomposition

$$\log \widehat{f}(\xi) = \log \widehat{f}_+(\xi) + \log \widehat{f}_-(\xi). \quad (25)$$

The conditions under which the above factorization or decomposition gives proper results have been given by Krein [33]; the most important requirement is that $\widehat{f}(\xi)$ is not zero anywhere.

In general neither the factorization nor the decomposition can be done analytically. With continuous monitoring, where the quantity to be factorized is $\phi(\xi, q) = 1 - q\psi(\xi)$ instead of $\Phi(\xi, q) = 1 - q\Psi(\xi, \Delta)$, an analytical treatment becomes possible for a Brownian motion or if we impose strong restrictions on the structure of the considered Lévy process [38, 45], such as the assumption that it is spectrally one-sided, i.e., jumps are either always up or always down. Another assumption that makes the factorization feasible is if the jumps are of phase type [4], which includes the Kou double exponential jump model [32] as a special case. In these cases the Wiener-Hopf factorization is tractable because $1 - q\psi(\xi)$ is a rational function and its decomposition in upper/lower factors is quite immediate. For example, Jeannin and Pistorius [28] approximate different Lévy models by the class of generalized hyper-exponential models, which have a tractable Wiener-Hopf factorization. A similar idea is pursued by Asmussen et al. [5] and Cai [9]. A quasi-analytical WH factorization has been achieved by Kuznetsov [34] when the characteristic exponent is a meromorphic function. In this case, WH factors can be expressed as infinite products and require the solution of transcendental equations. Unfortunately, with discrete monitoring even under the above assumptions the factorization is not doable analytically, because $1 - q\Psi(\xi, \Delta)$ is no more a rational function. In addition, all the above mentioned methods consider only the single-barrier case. An exception was given by Boyarchenko and Levendorskii [8], who obtained exact analytical pricing formulae in terms of Wiener-Hopf factors, and, under additional conditions on the process, derived simpler approximate formulae. For the general difficulty in computing the factors, with reference to the important financial engineering problem of pricing barrier options, Carr and Crosby [11] state: *“Pricing barrier options for arbitrary Lévy processes is far from trivial. There are, in principle, some results ... based on Wiener-Hopf analysis although they involve inversion of triple Laplace transforms and it is open to debate as to whether this could be done efficiently enough for use in a trading environment.”* Similarly, Cont and Tankov [14], a popular reference text for applications of Lévy processes in finance, states: *“The Wiener-Hopf technique is too computationally expensive and we recommend Monte Carlo simulation or numerical solution of partial integro-differential equations.”* These remarks are based on the representation of the Wiener-Hopf factors for the continuous-monitoring case as double integrals [14, Chapter 11.3]:

$$\begin{aligned}\log \varphi_+(\xi, q) &= \int_0^{+\infty} t^{-1} e^{-t/q} \int_0^{+\infty} (1 - e^{i\xi x}) p_X(x, t) dx dt \\ \log \varphi_-(\xi, q) &= \int_0^{+\infty} t^{-1} e^{-t/q} \int_{-\infty}^0 (1 - e^{i\xi x}) p_X(x, t) dx dt.\end{aligned}$$

With reference to financial applications, attempts to compute the Wiener-Hopf factors have been done by Boyarchenko and Levendorskii [8] and Kuznetsov et al. [35], among others.

A more convenient representation of the Wiener-Hopf factors can be found using the Hilbert transform and the Plemelj-Sokhotsky relations [31]. The Hilbert transform of a function $\widehat{f}(\xi)$ is defined as

$$\mathcal{H}_\xi \widehat{f}(\xi) = \text{p.v.} \frac{1}{\pi \xi} * \widehat{f}(\xi) = \text{p.v.} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\widehat{f}(\xi')}{\xi - \xi'} d\xi', \quad (26)$$

where $*$ denotes convolution and p.v. the Cauchy principal value,

$$\text{p.v.} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\widehat{f}(\xi')}{\xi - \xi'} d\xi' = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\pi} \left(\int_{\xi-1/\epsilon}^{\xi-\epsilon} \frac{\widehat{f}(\xi')}{\xi - \xi'} d\xi' + \int_{\xi+\epsilon}^{\xi+1/\epsilon} \frac{\widehat{f}(\xi')}{\xi - \xi'} d\xi' \right);$$

the latter assigns a value to an improper integral which would otherwise result in the indefinite form $+\infty - \infty$. The convolution theorem

$$\widehat{f}(\xi) * \widehat{g}(\xi) = \mathcal{F}[f(x)g(x)], \quad (27)$$

which maps a convolution to a product via a Fourier transform, together with the inverse Fourier transform

$$\text{p.v.} \mathcal{F}^{-1} \frac{1}{\pi \xi} = -i \operatorname{sgn} x,$$

enables to express the Hilbert transform through an inverse Fourier transform (from $\widehat{f}(\xi)$ to $f(x)$) and a direct Fourier transform,

$$i\mathcal{H}\widehat{f}(\xi) = \mathcal{F}[\operatorname{sgn} x f(x)]. \quad (28)$$

Thus a fast method to compute the Hilbert transform numerically consists simply in evaluating Equation (28) through an inverse and a direct FFT.

Define the projections of a function $f(x)$ on the positive or the negative half-axis through the multiplication with the indicator function of that set,

$$\mathcal{P}_x^+ f(x) := \mathbf{1}_{x>0} f(x) = f_+(x), \text{ and } \mathcal{P}_x^- f(x) := \mathbf{1}_{x<0} f(x) = f_-(x).$$

Now substitute

$$\operatorname{sgn} x f(x) = (\mathbf{1}_{x>0} - \mathbf{1}_{x<0}) f(x) = f_+(x) - f_-(x)$$

into Equation (28), obtaining the remarkable property

$$\widehat{f}_+(\xi) - \widehat{f}_-(\xi) = i\mathcal{H}\widehat{f}(\xi). \quad (29)$$

Together with the identity

$$\widehat{f}_+(\xi) + \widehat{f}_-(\xi) = \widehat{f}(\xi), \quad (30)$$

this allows to achieve a decomposition of a function $\widehat{f}(\xi)$, and thus a factorization of $\exp \widehat{f}(\xi)$, via its Hilbert transform. To this end, Equations (29) and (30) are conveniently rearranged to the Plemelj-Sokhotsky relations

$$\widehat{f}_+(\xi) = \frac{1}{2}[\widehat{f}(\xi) + i\mathcal{H}\widehat{f}(\xi)], \quad \widehat{f}_-(\xi) = \frac{1}{2}[\widehat{f}(\xi) - i\mathcal{H}\widehat{f}(\xi)]. \quad (31)$$

Obtaining the Wiener-Hopf factors of $\exp \widehat{f}(\xi)$ through Equations (31) with the Hilbert transform computed in a straightforward way by Equation (28) corresponds to performing in sequence an inverse Fourier transform, a projection on the positive or negative half axis and a Fourier transform,

$$\widehat{f}_+(\xi) = \mathcal{F}_{x \rightarrow \xi}[\mathcal{P}_x^+ \mathcal{F}_{\xi \rightarrow x}^{-1} \widehat{f}(\xi)](\xi), \quad \widehat{f}_-(\xi) = \mathcal{F}_{x \rightarrow \xi}[\mathcal{P}_x^- \mathcal{F}_{\xi \rightarrow x}^{-1} \widehat{f}(\xi)](\xi),$$

corresponding to the scheme

$$\widehat{f} \xrightarrow{\mathcal{F}^{-1}} f \xrightarrow{\mathcal{P}^+} f_+ \xrightarrow{\mathcal{F}} \widehat{f}_+ \quad \text{and} \quad \widehat{f} \xrightarrow{\mathcal{F}^{-1}} f \xrightarrow{\mathcal{P}^-} f_- \xrightarrow{\mathcal{F}} \widehat{f}_-.$$

This factorization is fast because it can be accomplished numerically with two FFTs and one projection [25, 44]. On the other hand, switching back and forth between Fourier and real space, the application of the projection causes a loss of accuracy; in the end this procedure turns out to have only quadratic accuracy.

A numerically more accurate approach consists in the computation of the Hilbert transform, and thus of the Plemelj-Sokhotsky relations, using a sinc expansion approximation to analytic functions. This approach uses two FFTs too to multiply Toeplitz matrices with vectors and thus has a computational cost of $\mathcal{O}(M \log M)$, but it does not leave Fourier space and its discretization error decreases exponentially with respect to M ; see Section 3.1 for details.

We stress here the similarities and differences with the approach followed by Feng and Linetsky [16, 17]. In the mentioned papers the Hilbert transform is applied in the backward-in-time pricing procedure. In practice, the projection step is performed in Fourier space using the Hilbert transform; greater details on how this is possible are given in the online supplementary material. This transform is computed at a high degree of accuracy via sinc expansion. No direct relationship of their procedure with a Wiener-Hopf factorization can be devised. The analogy is that we are able to express Wiener-Hopf factors via a Hilbert transform and then we can exploit their

idea of performing this transform with a sinc expansion. At the end, we are able to achieve the same accuracy as their method, but with a significant saving of computational time, because our procedure has a cost independent of the number of monitoring dates N , whilst in all existing methods, including the one by Feng and Linetsky [16], the cost increases linearly with N .

For the sake of truth, an advantage of the Feng and Linetsky method with respect to our procedure is that like all backward-in-time recursive methods it can easily deal with non-equally spaced monitoring dates. On the other side, our methodology can cope with the continuous monitoring case, as shown in the online supplementary material, whilst the Feng and Linetsky approach, and other Fourier methods, cannot.

The new approach proposed in the present paper is therefore summarized in the following procedure: 1) we perform the Wiener-Hopf factorization through the Plemelj-Sokhotsky relations (31), and 2) we compute the Hilbert transform in Fourier space using sinc functions as described in detail in the next section. The inversion of the z -transform is performed exploiting the Euler acceleration technique. The detailed procedure is discussed in Section 3. Applications to different exotic options are considered in Section 4.

3. Discrete approximation error and efficient implementation

The implementation of the proposed procedure to estimate the distributions in Equations (3)–(8) consists of two steps: an efficient implementation of the WH factorization exploiting sinc functions, and the z -transform inversion combined with Euler summation. The numerical implementation is detailed in this section.

3.1. Hilbert transform with sinc functions

The Hilbert transform can be efficiently computed using the sinc expansion approximation of analytic functions. The use of sinc functions,

$$S_k(z, h) = \frac{\sin(\pi(z - kh)/h)}{\pi(z - kh)/h}, \quad k \in \mathbb{Z},$$

has been deeply studied by Stenger [49, 50], who showed that a function $f(z)$ analytic on the whole complex plane and of exponential type with parameter π/h , i.e., $|f(z)| \leq Ce^{\pi|z|/h}$, can be reconstructed exactly from the knowledge of its values on an equispaced grid of step h , as $f(z)$ admits the sinc expansion [49, Theorem 1.10.1]

$$f(z) = \sum_{k=-\infty}^{+\infty} f(kh)S_k(z, h).$$

Now, since $\mathcal{F}_{z \rightarrow \zeta} S_k(z, h) = h e^{ikh\zeta}$, and [16, Corollary 6.1]

$$\mathcal{H}_z S_k(z, h) = \frac{1 - \cos(\pi(z - kh)/h)}{\pi(z - kh)/h},$$

also the Fourier and Hilbert transforms of $f(z)$ admit the sinc expansions

$$\widehat{f}(\zeta) = h \sum_{k=-\infty}^{+\infty} f(kh) e^{ikh\zeta} \quad \text{if } |\zeta| < \pi/h,$$

$\widehat{f}(\zeta) = 0$ if $|\zeta| \geq \pi/h$, since functions analytic on the whole plane and of exponential type have Fourier transforms that vanish outside of the finite interval $(-\pi/h, \pi/h)$ [49, Theorem 1.10.1], and

$$\mathcal{H}f(z) = \sum_{k=-\infty}^{+\infty} f(kh) \frac{1 - \cos(\pi(z - kh)/h)}{\pi(z - kh)/h}.$$

The integrals of f and $|f|^2$ can be written as sinc expansions too,

$$\int_{\mathbb{R}} f(x) dx = h \sum_{k=-\infty}^{+\infty} f(kh), \quad \int_{\mathbb{R}} |f(x)|^2 dx = h \sum_{k=-\infty}^{+\infty} |f(kh)|^2.$$

The above results show in particular that the trapezoidal quadrature rule with step size h is exact.

This holds true for a function $f(z)$ that is analytic in the whole complex plane. However, this can be used also to approximate a function that is analytic only in a strip including the real axis, which is the case considered in this article. More precisely, Stenger [49, Theorems 3.1.3, 3.1.4 and 3.2.1] states that in this case the trapezoidal approximation has an error that decays exponentially with respect to h .

The computation of the Hilbert transform via a sinc expansion can be performed using the FFT [16, Section 6.5]. A discrete Hilbert transform requires matrix-vector multiplications involving Toeplitz matrices. As is well known, this kind of multiplications can be performed exploiting the FFT, once those matrices are embedded in a circulant matrix [16, Appendix B] [20]. In particular, Feng and Linetsky, with respect to the computation of the Hilbert transform [17, Theorem 3.3] and of the whole Plemelj-Sokhotsky formulas (31) [16, Theorem 6.5] [17, Theorem 3.4] with sinc functions, proved the following convergence result: if a function is analytic in a suitable strip around the real axis, then the discretization error of its numerical factorization or decomposition decays exponentially with the number of discretization points M . Matlab code to perform the Hilbert transform via sinc functions and therefore the WH factorization is provided in the online supplementary material.

3.2. Acceleration of the inverse z -transform via Euler summation

In order to recover the probabilities in Equations (11)–(16) once the WH equations have been solved, we need an inverse z -transform $\mathcal{Z}_{q \rightarrow n}^{-1}$. The latter is performed according to Equation (23), where $\rho \in (0, 1)$ is a free parameter; setting $\rho = 10^{-6}$ yields a 10^{-12} accuracy of the option price [1, 20]. Moreover, we apply the Euler summation, which is a convergence-acceleration technique well suited to evaluate alternating series. The idea of the Euler summation is to approximate $\mathcal{Z}_{q \rightarrow n}^{-1} \tilde{v}(\xi, q)$ by the binomial average, also called Euler transform, of its partial sums b_k from $k = n_E$ to $k = n_E + m_E$, i.e.,

$$\mathcal{Z}_{q \rightarrow n}^{-1} \tilde{v}(\xi, q) \approx \frac{1}{2^{m_E} n \rho^n} \sum_{j=0}^{m_E} \binom{m_E}{j} b_{n_E+j}(\xi), \quad (32)$$

where

$$b_k = \sum_{j=0}^k (-1)^j a_j \operatorname{Re} \tilde{v}(\xi, \rho e^{ij\pi/n}),$$

with $a_0 = 0.5$, $a_j = 1$, $j = 1, \dots, n_E + m_E$, and n_E and m_E are suitably chosen such that $n_E + m_E < n$. Thus the number of parameters $q = \rho e^{ij\pi/n}$ to be considered in Equation (23) drops from $n+1$ to $n_E + m_E + 1$. Numerical tests suggest to set $n_E = 12$ and $m_E = 20$.

In conclusion, the combined use of the numerical Hilbert transform and the numerical inverse z -transform has a computational cost of

$$\mathcal{O}((\min\{n, n_E + m_E\} + 1)M \log M),$$

and a discretization error which exponentially decays till it reaches an accuracy of about 10^{-12} . This is confirmed in the numerical experiments reported in Section 5 to price derivatives. The only exception is for the double-barrier case, and therefore when we deal with the probability $dP_{X,m,M}$, where the decay of the error turns out to be only polynomial, due to the use of the fixed-point algorithm. On the other hand, the iterative numerical scheme solves a long-standing problem related to an efficient computation of the Wiener-Hopf factors in the double-barrier case.

4. Applications to option pricing

In mathematical finance Lévy processes $X(t)$ are used to describe the evolution of an asset price $S(t)$ according to

$$S(t) = S_0 e^{X(t)},$$

$S_0 = S(0)$ being the initial spot price. The stock price dynamics is directly specified under the so-called risk-neutral measure, so that in Equation (1) $a = r - \delta - \frac{1}{2}\sigma^2 - \int_{\mathbb{R}} (e^\eta - 1 - \eta \mathbf{1}_{|\eta| < 1}) \nu(d\eta)$, where r is the risk-free interest rate and δ the asset dividend rate.

To price path-dependent options such as barrier and lookback options, the relevant quantities are the maximum M_N and the minimum m_N registered at discrete times $t = n\Delta$, $n = 0, \dots, N$, up to maturity $N\Delta = T$, Δ being the constant time interval between two subsequent monitoring dates. For a fixed-strike lookback option we need the distribution $P_M(x, N)$ of the maximum or $P_m(x, N)$ of the minimum. For a single-barrier option we need the joint distribution $P_{X,M}(x, N)$ or $P_{X,m}(x, N)$ of the Lévy process at T and of its maximum (up-and-out case) or minimum (down-and-out case) over all monitoring dates $n = 0, \dots, N$. For a double-barrier option we need the joint distribution $P_{X,m,M}(x, N)$ of the triplet (X_N, m_N, M_N) .

In pricing the above mentioned contracts, we are interested in the truncated damped payoff

$$\phi(x) = e^{\alpha x} S_0 (e^x - e^k)^+ \mathbf{1}_{x \leq u} \quad (33)$$

for a call option and

$$\phi(x) = e^{\alpha x} S_0 (e^k - e^x)^+ \mathbf{1}_{x \geq l} \quad (34)$$

for a put option, where $k = \log(K/S_0)$ is the rescaled log-strike of the option, and $l = \log(L/S_0)$ and $u = \log(U/S_0)$ are the rescaled lower and upper log-barriers. In the following we assume $l < k < u$. The damping factor $e^{\alpha x}$ with a suitable choice of the parameter α makes the Fourier transform of the payoff well defined.

The option price is obtained discounting the expectation value of the undamped payoff with respect to the appropriate distribution; this expectation can conveniently be computed through the Parseval/Plancherel relation [39] by a product in Fourier space and an inverse Fourier transform,

$$\begin{aligned} \mathbb{E}[\phi(x)e^{-\alpha x}] &= \int_{-\infty}^{+\infty} \phi(x)e^{-\alpha x} p(x) dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{\phi}(\xi) \widehat{p}^*(\xi + i\alpha) d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{\phi}^*(\xi) \widehat{p}(\xi + i\alpha) d\xi = \mathcal{F}_{\xi \rightarrow x}^{-1} \left[\widehat{\phi}^*(\xi) \widehat{p}(\xi + i\alpha) \right] (0), \end{aligned} \quad (35)$$

where $p(x) = p_M(x, N)$ or $p_m(x, N)$ for lookback options (to be synthetic, in the following we will consider only fixed-strike lookback options written on the minimum), $p(x) = p_{X,M}(x, N)$ or $p_{X,m}(x, N)$ for single-barrier options, and $p = p_{X,m,M}(x, N)$ for double-barrier options. The introduction

of a damping factor in the payoff is compensated by a shift of the Fourier transform of the probability density function.

While it is known that the Fourier transform of the truncated damped payoff for a barrier option is

$$\widehat{\phi}(\xi) = K e^{k(\alpha+i\xi)} \left(\frac{1 - e^{b(\alpha+i\xi)}}{\alpha + i\xi} - \frac{1 - e^{b(1+\alpha+i\xi)}}{1 + \alpha + i\xi} \right) \quad (36)$$

with $b = \log(U/K)$ for a call option and $b = \log(L/K)$ for a put option [16], the main problem in evaluating path-dependent options is the computation of the characteristic functions of the (joint) probability densities defined in Equations (5)–(8). Here we exploit the Spitzer identity and the factorization procedure previously described. So let us assume for the moment that the quantities appearing on the right-hand side of Equations (12)–(16) are known; then if we take the z -transform of the undiscounted expectation value in Equation (35), we obtain

$$\sum_{n=0}^{\infty} q^n \mathbb{E} [\phi(x) e^{-\alpha x}] = \sum_{n=0}^{\infty} q^n \mathcal{F}^{-1} [\widehat{\phi}^*(\xi) \widehat{p}(\xi + i\alpha, n)](0),$$

and swapping the z -transform with the Fourier transform

$$\mathcal{F}_{\xi \rightarrow x}^{-1} \left[\widehat{\phi}^*(\xi + i\alpha) \sum_{n=0}^{\infty} q^n \widehat{p}(\xi + i\alpha, n) \right] (0) = \mathcal{F}_{\xi \rightarrow x}^{-1} \left[\widehat{\phi}^*(\xi) \widetilde{p}(\xi + i\alpha, q) \right] (0).$$

The interchange of integration and summation requires the z -transform $\widetilde{f}(z, q) = \sum_{n=0}^{\infty} q^n f(z, n)$ to converge uniformly. The z -transform is in fact a power series in q with coefficients $f(z, n)$ and radius of convergence given by ρ . A power series converges uniformly in a closed and bounded interval contained in the interval of convergence [18]. Using the inverse z -transform defined in Equation (9) we finally obtain the option price through the double inverse transform

$$v(x, n) = e^{-rT} \mathcal{F}_{\xi \rightarrow x}^{-1} \left[\widehat{\phi}^*(\xi) \mathcal{Z}_{q \rightarrow n}^{-1} \left[\widetilde{p}(\xi + i\alpha, q) \right] \right] (x, n), \quad (37)$$

evaluated for $(x, n) = (0, N)$. A few little improvements, discussed later on, can be adopted in order to enhance the numerical accuracy of the final result.

In Equation (37) the inverse z -transform is performed before the inverse Fourier transform to minimize the computational cost. The reason is that the inversion operator $\mathcal{Z}_{q \rightarrow n}^{-1}$ is well approximated by a sum of $n + 1$ terms (or $n_E + m_E + 1$ if the Euler acceleration is considered). Therefore, from a

computational point of view it is advantageous to do a single inverse Fourier transform of the sum instead of a separate transform of each of the addends.

Since we have to deal with unbounded domains, we use a domain truncation based on a moments bound with tolerance 10^{-8} [20]; thus the truncation error is constant, but, according to the reported numerical experiments [19, 20], it does not affect the first significant decimal digits.

4.1. Lookback options

In the case of lookback options (to be synthetic, we deal only with a fixed-strike lookback on the minimum), assuming a number of monitoring dates $N > 1$, it is convenient to modify the pricing formula (37) into

$$v(x, N) = e^{-rT} \mathcal{F}_{\xi \rightarrow x}^{-1} \left[\Psi(\xi + i\alpha) \widehat{\phi}^*(\xi) \mathcal{Z}_{q \rightarrow N-1}^{-1} \left[\frac{1}{\Phi_+(0, q) \Phi_-(\xi + i\alpha, q)} \right] \right],$$

evaluated for $x = 0$, i.e., to apply the inverse z -transform to a number of monitoring dates reduced by 1, and to account for the extra date multiplying the conjugated Fourier transform of the payoff function by the characteristic function Ψ . This smooths the payoff function, giving it the required regularity to ensure an exponential decay of the error. From a financial point of view, this smoothing is equivalent to price a lookback option with $N - 1$ monitoring dates and a payoff $v_1 = v(x, 1)$, where $v(x, n)$ is the value of the option at time $(N - n)\Delta$ and log-price x . Thus, lookback options are priced according to the algorithm

$$\phi \equiv v_0 \xrightarrow{\mathcal{F}} \widehat{v}_0 \xrightarrow{\Psi^*} \widehat{v}_1 \xrightarrow{\mathcal{ZS}} \widehat{v}_N \xrightarrow{\mathcal{F}^{-1}} v_N,$$

where the operator \mathcal{ZS} is defined as

$$\mathcal{ZS} : F(\xi) \rightarrow F^*(\xi) \mathcal{Z}_{q \rightarrow N-1}^{-1} \left[\frac{1}{\Phi_+(0, q) \Phi_-(\xi, q)} \right].$$

Recall that the conjugate operator applied to the generic function F is due to the Parseval relation. The full procedure consists of the following steps:

1. For each q necessary to invert the z -transform, factorize

$$\Phi(\xi, q) := 1 - q\Psi(\xi, q) = \Phi_+(\xi, q) \Phi_-(\xi, q)$$

and compute the Spitzer identity

$$\widetilde{p}_m(\xi, q) = \frac{1}{\Phi_+(0, q) \Phi_-(\xi, q)}.$$

2. Apply the inverse z -transform $\mathcal{Z}_{q \rightarrow N-1}^{-1}$ to $\widetilde{p}_m(\xi + i\alpha, q)$ and multiply the result by $\Psi(\xi + i\alpha)\widehat{\phi}^*(\xi)$, obtaining $\widehat{v}(\xi, N)$.
3. Apply the inverse FFT to $\widehat{v}(\xi, N)$ and pick the value for $x = 0$, obtaining the option price.

A similar procedure is valid for fixed-strike lookback options written on the maximum, where $\widetilde{p}_M(\xi, q)$ is used in place of $\widetilde{p}_m(\xi, q)$.

4.2. Single-barrier options

Without loss of generality, let us consider the case of a down-and-out barrier option. Assuming a number of monitoring dates $N > 2$, we reduce this number by 1 and multiply the payoff function by the characteristic function, as we did for lookback options. Then our methodology is applied to the remaining $N - 1$ monitoring dates. The algorithm can be summarized as

$$\phi \equiv v_0 \xrightarrow{\mathcal{F}} \widehat{v}_0 \xrightarrow{\Psi^*} \widehat{v}_1 \xrightarrow{\mathcal{ZS}} \widehat{v}_N \xrightarrow{\mathcal{F}^{-1}} v_N \quad (38)$$

$$\delta \xrightarrow{\mathcal{F}} \cdot \xrightarrow{\Psi} \uparrow \quad (39)$$

In this case we denote with \mathcal{ZS} the operator

$$\mathcal{ZS} : F(\xi) \rightarrow F^*(\xi) \mathcal{Z}_{q \rightarrow N-2}^{-1} \left[e^{il\xi} \frac{\overline{P}_+(\xi, q)}{\Phi_+(\xi, q)} \right],$$

where

$$\overline{P}(\xi, q) := \Psi(\xi) \frac{e^{-il\xi}}{\Phi_-(\xi, q)} = \overline{P}_+(\xi, q) + \overline{P}_-(\xi, q).$$

If we compare the above operator with the definition of $\widetilde{p}_{X,m}$ in Equation (14), we notice that the only difference is the presence of \overline{P} , which differs from P only by the factor Ψ . This is necessary to smooth the function P , substituting it with \overline{P} , that has the regularity required to ensure an exponential decay of the error. The substitution is related to the procedure sketched in Equation (39): in computing the distribution $P_{X,m}$ we do not start from time 0, but we move one step forward with a convolution procedure, which corresponds to multiplying with Ψ the Fourier transform of the Dirac delta function, i.e., the value of the probability at time 0, and then apply the Spitzer identity. Moreover, notice that the procedures given by Equation (38) and Equation (39) are performed backward and forward-in-time, respectively, since the first one is related to the price of the derivative

(starting point: payoff at time T), while the second one to the probability distribution of the log-price (starting point: Dirac delta at time 0)².

Therefore, for a down-and-out barrier option we perform the following steps:

1. For each q necessary to invert the z -transform, factorize

$$\Phi(\xi, q) := 1 - q\Psi(\xi, q) = \Phi_+(\xi, q)\Phi_-(\xi, q),$$

decompose

$$\overline{P}(\xi, q) := \Psi(\xi) \frac{e^{-i\xi}}{\Phi_-(\xi, q)} = \overline{P}_+(\xi, q) + \overline{P}_-(\xi, q),$$

and compute the Spitzer identity

$$R(\xi, q) := e^{i\xi} \frac{\overline{P}_+(\xi, q)}{\Phi_+(\xi, q)}. \quad (40)$$

The function $R(\xi, q)$ is related to $\widetilde{\widehat{p}}_{X,m}(\xi, q)$ in Equation (12): more precisely, $\mathcal{Z}_{q \rightarrow N-1}^{-1} R(\xi, q) = \mathcal{Z}_{q \rightarrow N}^{-1} \widetilde{\widehat{p}}_{X,m}(\xi, q) = \widehat{p}_{X,m}(\xi, N)$.

2. Apply the inverse z -transform $\mathcal{Z}_{q \rightarrow N-2}^{-1}$ and then the inverse Fourier transform, obtaining the option price from

$$v(x, N) = e^{-rT} \mathcal{F}_{\xi \rightarrow x}^{-1} \left[\Psi(\xi + i\alpha) \widehat{\phi}^*(\xi) \mathcal{Z}_{q \rightarrow N-2}^{-1} \left[e^{i\xi(\xi+i\alpha)} \frac{\overline{P}_+(\xi + i\alpha, q)}{\Phi_+(\xi + i\alpha, q)} \right] \right]$$

evaluated for $x = 0$, where $\widehat{\phi}^*$ is the conjugated Fourier transform of the payoff, Equation (36), and conjugated because of its use within the Parseval relation, Equation (35).

4.3. Double-barrier options

For the double-barrier option pricing problem the missing piece is the computation of the factors J_+ and J_- in Equation (16). This requires the solution of a system of two integral equations, and we apply here the new fixed-point algorithm presented in Section 2.

Starting from Equations (17)–(18), as for the single-barrier case we assume a number of monitoring dates $N > 2$ and we move one step forward

²We recall that the Fourier transform of the backward-in-time transition density is $\Psi^*(\xi, \Delta) = \mathcal{F}_{x \rightarrow \xi}[f(-x, \Delta)](\xi, \Delta)$, where $f(x, \Delta)$ is the forward-in-time transition density and $\Psi(\xi, \Delta)$ its Fourier transform.

in the computation of the probability $dP_{X,m,M}$ via convolution. This corresponds to consider

$$\frac{J_-(\xi, q)}{\Phi_-(\xi, q)} = \left[\frac{e^{-il\xi}\Psi(\xi) - e^{i(u-l)\xi}J_+(\xi, q)}{\Phi_-(\xi, q)} \right]_-, \quad (41)$$

$$\frac{J_+(\xi, q)}{\Phi_+(\xi, q)} = \left[\frac{e^{-iu\xi}\Psi(\xi) - e^{i(l-u)\xi}J_-(\xi, q)}{\Phi_+(\xi, q)} \right]_+, \quad (42)$$

instead of Equations (19)–(20). To compute J_\pm we consider the iterative procedure presented in Section 2, dealing with

$$\overline{P}^{(j)}(\xi, q) := \frac{e^{-il\xi}\Psi(\xi)}{\Phi_-(\xi, q)} - \frac{e^{i(u-l)\xi}J_+^{(j-1)}(\xi, q)}{\Phi_-(\xi, q)} = \overline{P}_+^{(j)}(\xi, q) + \overline{P}_-^{(j)}(\xi, q)$$

and

$$\overline{Q}^{(j)}(\xi, q) := \frac{e^{-iu\xi}\Psi(\xi)}{\Phi_+(\xi, q)} - \frac{e^{i(l-u)\xi}J_-^{(j)}(\xi, q)}{\Phi_+(\xi, q)} = \overline{Q}_+^{(j)}(\xi, q) + \overline{Q}_-^{(j)}(\xi, q)$$

instead of Equations (21)–(22), respectively. Once J_\pm are obtained via the fixed-point algorithm, we compute

$$R(\xi, q) := \frac{\Psi(\xi)}{\Phi(\xi, q)} - e^{il\xi} \frac{J_-(\xi, q)}{\Phi(\xi, q)} - e^{iu\xi} \frac{J_+(\xi, q)}{\Phi(\xi, q)}. \quad (43)$$

The function $R(\xi, q)$ is related to $\widetilde{\widehat{p}}_{X,m,M}(\xi, q)$ in Equation (16): more precisely, $\mathcal{Z}_{q \rightarrow N-1}^{-1} R(\xi, q) = \mathcal{Z}_{q \rightarrow N}^{-1} \widetilde{\widehat{p}}_{X,m,M}(\xi, q) = \widehat{p}_{X,m,M}(\xi, N)$.

Therefore the scheme for the computation of the option price is the following:

1. For each q necessary to invert the z -transform, factorize

$$\Phi(\xi, q) = 1 - q\Psi(\xi, q) = \Phi_+(\xi, q)\Phi_-(\xi, q),$$

and compute $R(\xi, q)$ via the iterative scheme.

2. Apply the inverse z -transform $\mathcal{Z}_{q \rightarrow N-2}^{-1}$ to $R(\xi + i\alpha, q)$ and then the inverse Fourier transform, obtaining the option price from

$$v(x, N) = e^{-rT} \mathcal{F}_{\xi \rightarrow x}^{-1} \left[\Psi(\xi + i\alpha) \widehat{\phi}^*(\xi) \mathcal{Z}_{q \rightarrow N-2}^{-1} [R(\xi + i\alpha, q)] \right] \quad (44)$$

evaluated for $x = 0$. Thus, the methodology to price a double-barrier option is close to the one proposed for single-barrier contracts and consists of the same steps as sketched in Equations (38)–(39), with a different $R(\xi, q)$ inside

the operator $\mathcal{ZS} : F(\xi) \rightarrow F^*(\xi) \mathcal{Z}_{q \rightarrow N-2}^{-1} [R(\xi, q)]$, i.e. the $R(\xi, q)$ computed via the fixed-point algorithm in Equation (43) instead of the one defined in Equation (40). Even if the factorization is performed with a Hilbert transform computed as proposed in Section 3.1, our numerical experiments show that this pricing algorithm provides a quadratic convergence of the error, instead of the exponential one of single-barrier (and lookback) options.

5. Numerical experiments

In this section we compare the proposed pricing techniques with others presented in the literature. We consider:

- Z-S, i.e., the new fast method presented in this article.
- CONV, i.e., the convolution method of Lord et al. [42] described in the supplementary material, Section C.1.
- HILB, i.e., the recursive method of Feng and Linetsky [16] based on the Hilbert transform and described in the supplementary material, Section C.1.
- REC-QUAD, i.e., the recursive method based on the trapezoidal quadrature rule and described in the supplementary material, Section C.2.
- Z-QUAD, i.e., the method of Fusai et al. [20] based on the z -transform and the trapezoidal quadrature rule, described in the supplementary material, Section C.3.

The Z-QUAD algorithm requires to solve several WH integral equations via quadrature formulas. Another possibility consists in relating the Spitzer-WH factorization to the solution of these integral equations. Indeed, the well-known methodology to solve a WH integral equation also requires the knowledge of the WH factors. Therefore, we also consider the following new method

- Z-WH, i.e., a new method which improves Z-QUAD exploiting Wiener-Hopf factorization via the Hilbert transform and sinc functions; see the supplementary material, Section C.3.

All the numerical experiments have been performed with Matlab R2013b running under Windows 7 on a personal computer equipped with an Intel Core i7 Q720 1600 MHz processor and 6 GB of RAM. We would like to stress that with lookback and single-barrier options and with all Fourier-based methods we have unbounded domains. Therefore, as already stated in

Figure 1: Down-and-out barrier call option: pointwise absolute error as a function of the number of grid points M for $N = 100$ (left) and $N = 252$ (right) monitoring dates.

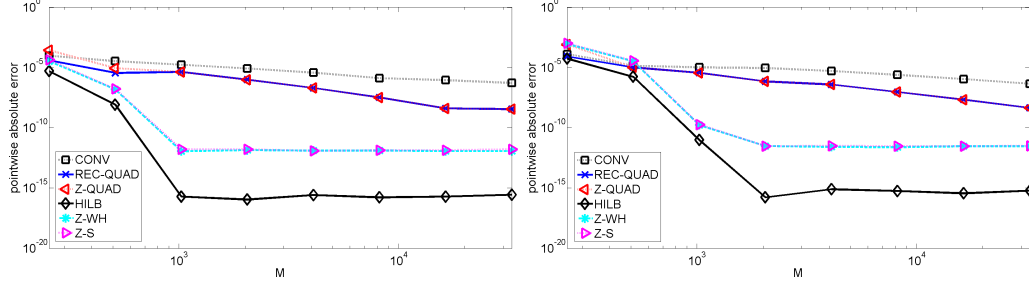
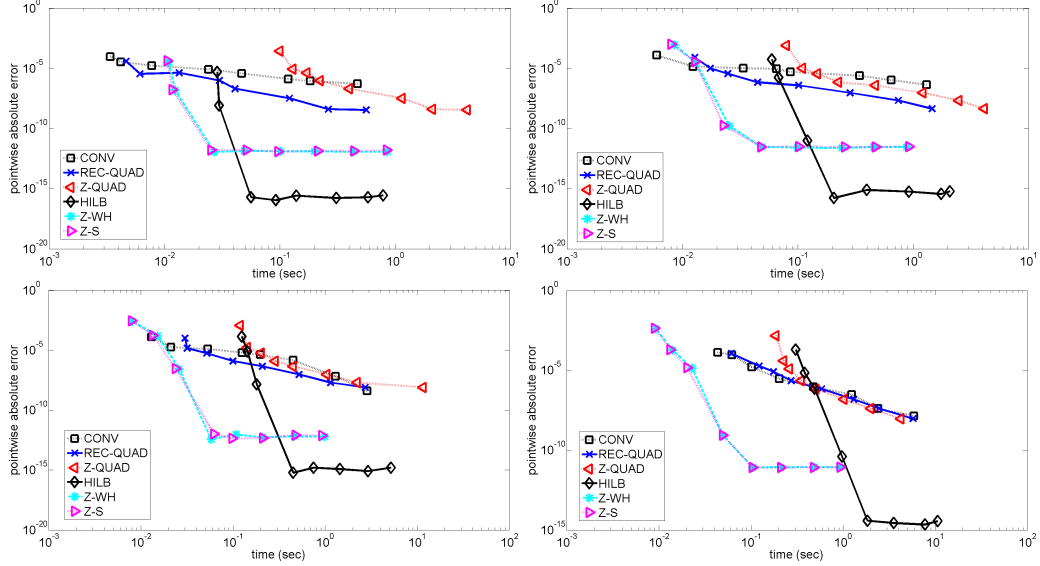


Figure 2: Down-and-out barrier call option: pointwise absolute error as a function of CPU time for $N = 100$ (top left), $N = 252$ (top right), $N = 504$ (bottom left) and $N = 1260$ (bottom right) monitoring dates.



Section 4, we truncate the domain with a Chernoff bound computed according to the first ten moments, as suggested by Fusai et al. [20].

First of all, we consider a down-and-out call barrier option assuming that the underlying asset evolves according to a Merton jump diffusion process with the same parameters as in Feng and Linetsky [16], including the procedure to choose the damping parameter α . The lower barrier is $L = 0.8$, the initial spot price S_0 and the strike price K are both set to 1, and the time to maturity is $T = 1$. The underlying asset has a dividend rate $\delta = 0.02$ and the risk-free interest rate is $r = 0.05$.

Table 1: Down-and-out barrier call option: option price and CPU time in seconds; $M = 2^{14}$.

N	Z-S		Z-WH		HILB	
	Price	CPU time	Price	CPU time	Price	CPU time
50	0.04775954751	0.604597	0.04775954751	0.615977	0.04775954750	0.411529
100	0.04775180473	0.598856	0.04775180473	0.585755	0.04775180472	0.719666
252	0.04774580616	0.613833	0.04774580616	0.600996	0.04774580615	1.745266
504	0.04774337792	0.601078	0.04774337791	0.591950	0.04774337791	3.468807

In Figure 1 we consider the case with $N = 100$ and $N = 252$ monitoring dates: we report in double logarithmic scale the pointwise absolute error, computed at the spot price $S_0 = 1$, taking for reference as the exact solution the price computed with the HILB method and a grid of 2^{16} points. The CONV, REC and Z-QUAD methods have a polynomial convergence; moreover the REC and the Z-QUAD algorithms show a similar polynomial accuracy. Our newly proposed methods, Z-S and Z-WH, and the HILB algorithm exhibit an exponential convergence due to the use of the sinc expansion and to the fact that all computations are performed in Fourier space, as already described by Feng and Linetsky [16]. As expected, both the Z-S and Z-WH methods rapidly reach the maximum accuracy allowed by the approximation used to invert the z -transform, i.e., 10^{-12} .

In Figure 2 we report the pointwise absolute error against the CPU time necessary for the price computation for different numbers of monitoring dates. It is clear that the Z-S, the Z-WH and the HILB methods are the most accurate. Their exponential convergence enables them to be used with a limited number M of grid nodes. The Z-S and the Z-WH methods are able to compute option prices with an accuracy of 10^{-12} in less than a quarter of a second. Notice that increasing the number of monitoring dates from 252 to 504 or 1260, the computational costs of the methods based on the z -transform do not change because of the Euler acceleration technique. From these experiments it appears that, among the methods proposed in this paper, Z-S and Z-WH are preferable when the number of dates is large. However, if a greater accuracy is necessary and the number of monitoring dates is not too large, the HILB method by Feng and Linetsky [16] should also be considered.

To complete the numerical tests on single-barrier options, Table 1 shows results for a down-and-out barrier call option, assuming that the underlying asset evolves according to a NIG process with the same parameters as in Feng and Linetsky [16]. All the other parameters are as before. These results confirm the good performance of the Z-S and Z-WH algorithms when the number of monitoring dates increases.

Figure 3: Knock-and-out barrier call option: pointwise absolute error with $N = 252$.

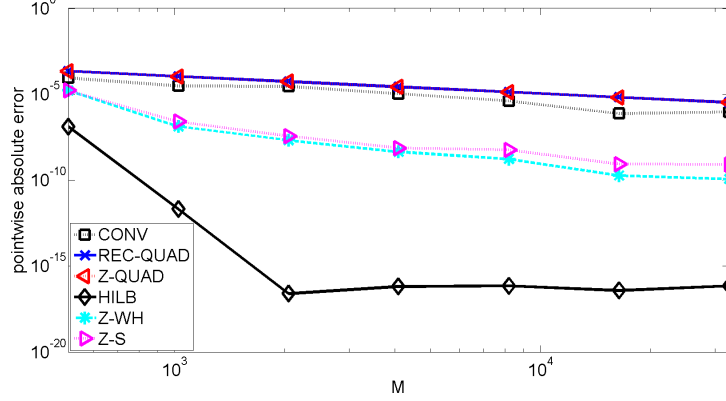


Table 2: Fixed-strike lookback call (on the maximum) and put (on the minimum) options: option price and CPU time in seconds.

M	Call		Put	
	Price	CPU time	Price	CPU time
2^8	0.183264603755	0.0097	0.117871584305	0.0087
2^9	0.183264598264	0.0169	0.117871585215	0.0114
2^{10}	0.183264598276	0.0214	0.117871585217	0.0175
2^{11}	0.183264598268	0.0361	0.117871585212	0.0371
2^{12}	0.183264598273	0.0722	0.117871585216	0.0964
2^{13}	0.183264598262	0.1933	0.117871585210	0.1753
2^{14}	0.183264598287	0.3211	0.117871585214	0.3052
2^{15}	0.183264598282	0.6192	0.117871585214	0.5601
2^{16}	0.183264598276	1.2780	0.117871585214	1.0442

In Figure 3 we consider a double-barrier option and we plot the pointwise absolute error for the fixed-point algorithm presented in Section 4.3. We use the Kou double exponential model [32], again with the same parameters as in Feng and Linetsky [16]. The lower (upper) barrier is $L = 0.8$ ($U = 1.2$), the initial spot price is $S_0 = 1$ and the strike price is $K = 1.1$. A one year daily monitoring is assumed, i.e., $T = 1$ and $N = 252$. The error is again computed considering as exact the solution computed with the HILB method and $M = 2^{16}$ grid points. The numerical experiments show that the orders of convergence of the newly proposed algorithms, Z-WH and Z-S, are no more exponential as in the single-barrier case, but approximately quadratic. We would like to stress that the average number of fixed-point iterations necessary to reach a tolerance of 10^{-12} is as low as 3. Moreover, the newly proposed methods are still slightly more accurate than the CONV, REC and Z-QUAD ones.

Finally, in Table 2 we price a fixed-strike lookback call option written on the maximum and a put option written on the minimum, both with $N = 50$ monitoring dates and $S_0 = K = 1$. We assume that the underlying asset evolves as a geometric Brownian motion with the same parameters as in Feng and Linetsky [17], i.e., $\sigma = 0.3$, $r = 0.1$, $T = 0.5$. We report the option price and the computational cost of the Z-S approach for different numbers of grid points M . For the call option, compare the benchmark price 0.183264598300 provided by Feng and Linetsky [17, Table 1]. From this table we notice the same exponential convergence of the algorithm as in the single-barrier case.

6. Conclusions

In this article we presented a fast and accurate constructive procedure to perform the Wiener-Hopf factorization of a complex function. As a concrete application we considered the pricing of barrier and lookback options, when the monitoring is discrete and the underlying evolves according to an exponential Lévy process. Our procedure is based on the combined use of Hilbert and z -transforms. The numerical implementation exploits the fast Fourier transform and the Euler summation. The computational cost is independent of the number of monitoring dates. In addition, the error decays exponentially with the number of grid points. For the double-barrier case we also introduce a new iterative algorithm based on the Wiener-Hopf factorization. Applications to pricing of exotic derivatives confirm the exponential accuracy of the proposed method. Extensions to other exotic derivatives, like perpetual Bermudan, occupation time, quantile and step options are straightforward combining our method with the Wendel-Port-Dassios identity [15]. Applications of the proposed procedure to other fields such as insurance, queuing theory and fluid mechanics are currently under investigation.

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